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# The algebraic connectivity of lollipop graphs ☆

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ABSTRACT

Let $C_{n,g}$ be the lollipop graph obtained by appending a $g$-cycle $C_g$ to a pendant vertex of a path on $n-g$ vertices. In 2002, Fallat, Kirkland and Pati proved that for $n \geq \frac{3g-1}{2}$ and $g \geq 4$, $\alpha(C_{n,g}) > \alpha(C_{n,g-1})$. In this paper, we prove that for $g \geq 4$, $\alpha(C_{n,g}) > \alpha(C_{n,g-1})$ for all $n$, where $\alpha(C_{n,g})$ is the algebraic connectivity of $C_{n,g}$.



## 1. Introduction

Let $G=(V,E)$ be a simple graph with vertex set $V=\{v_1,v_2,\ldots,v_n\}$ and edge set $E$. Let $d(v_i)$ be the degree of the vertex $v_i \in V(G)$ $(i=1,2,\ldots,n)$, and $D=D(G)=diag(d(v_1),d(v_2),\ldots,d(v_n))$ be the diagonal matrix of vertex degrees. The *Laplacian matrix* $L(G)=D(G)-A(G)$ is the difference of $D(G)$ and the adjacency matrix $A(G)$. It is easy to see that $L(G)$ is a positive semidefinite symmetric matrix with the smallest eigenvalue 0 and the corresponding eigenvector is the all ones column vector, which is denoted by $e$. Denote its eigenvalues by

$$\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G) = 0,$$

which are always enumerated in non-increasing order and repeated according to their multiplicity. Fiedler [6] showed that the second smallest eigenvalue of $L(G)$ is 0 if and only if $G$ is disconnected.

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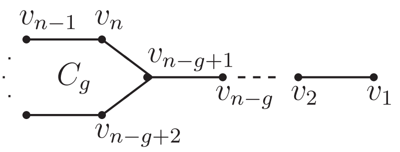


**Fig. 1**. The lollipop graph $C_{n,g}$.

Thus the second smallest eigenvalue of $L(G)$ is popularly known as the *algebraic connectivity* of $G$ and is usually denoted by $\alpha(G)$. Let $P_n$ and $C_n$ be the path and the cycle on $n$ vertices, respectively. It is a well-known fact that

$$\alpha(P_n) = 4\sin^2\frac{\pi}{2n} \text{ and } \alpha(C_n) = 4\sin^2\frac{\pi}{n}.$$

Let $Y \in \mathbb{R}^n$ be a column vector, and $Y(v)$ denote the entry of $Y$ corresponding to the vertex $v$ of $G$. Such labellings are sometimes called *valuations* of the vertices of $G$ (see [21]). If $X$ is a unit eigenvector of $G$ corresponding to $\alpha(G)$, we commonly call it a *"Fiedler vector"* of $G$. It is obvious that $X^T e = 0$ and

$$\alpha(G) = X^T L(G) X = \sum_{v_i v_j \in E} (X(v_i) - X(v_j))^2 = \min_{\substack{Y \in \mathbb{R}^n \setminus \{0\} \\ Y^T e = 0}} \frac{Y^T L(G) Y}{Y^T Y}. \tag{1.1}$$

The algebraic connectivity and Fiedler vectors have been well studied. Readers are referred to [1,2,6–9,16–20] for references on these topics.

The *lollipop graph* $C_{n,g}$ is obtained by appending a $g$-cycle $C_g$ to a pendant vertex of a path on $n-g$ vertices (see Fig. 1). In [14], the lollipop graph $C_{n,g}$ for odd $g$ is proved to be determined by its adjacency spectrum, and all lollipop graphs are proved to be determined by their Laplacian spectrum. In [22], it is also proved that the lollipop graphs are determined by their signless Laplacian spectrum. In [12], the first author proved that the lollipop graph $C_{n,g}$ has the least algebraic connectivity among all connected graphs with girth $g$. This confirms the conjecture proposed by Fallat and Kirkland [4]. Fallat et al. [5] proved that for $n \geq \frac{3g-1}{2}$ and $g \geq 4$, then $\alpha(C_{n,g}) > \alpha(C_{n,g-1})$. In this paper, we further obtain the following somewhat general result: if $g \geq 4$, then $\alpha(C_{n,g}) > \alpha(C_{n,g-1})$.

Throughout this paper, we shall denote by $\Phi(B) = \Phi(B; x) = \det(xI - B)$ the characteristic polynomial of the square matrix $B$. In particular, if $B = L(G)$, we write $\Phi(L(G))$ by $\Phi(G; x)$ or simply by $\Phi(G)$ and call $\Phi(G)$ the *Laplacian characteristic polynomial* of $G$.

## 2. Lemmas and results

Let $G$ be a graph and let $G' = G + e$ be the graph obtained from $G$ by inserting a new edge $e$ into $G$. The following lemma follows from the well-known Courant–Weyl inequalities (see e.g., [3, Theorem 2.1]).

**Lemma 2.1.** *The Laplacian eigenvalues of $G$ and $G'$ interlace, that is,*

$$\lambda_1(G') \geqslant \lambda_1(G) \geqslant \lambda_2(G') \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G') = \lambda_n(G) = 0$$

The following inequalities are known as Cauchy's inequalities and the whole theorem is also known as the interlacing theorem.

**Lemma 2.2** [15]. *Let $A$ be a Hermitian matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $B$ be a principal sub-matrix of order $m$; let $B$ have eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$. Then the inequalities $\lambda_{n-m+i} \leq \mu_i \leq \lambda_i$ $(i = 1, 2, \ldots, m)$ hold.*

**Lemma 2.3** [10]. *Let $G_1 = (V, E_1)$ be a graph on $n$ vertices and $G_2 = (V, E_2)$ be a graph obtained from $G_1$ by removing an edge and adding a new edge that was not there before. Then*

$$\lambda_i(G_1) \geq \lambda_{i+1}(G_2) \text{ and } \lambda_i(G_2) \geq \lambda_{i+1}(G_1) \text{ for } 1 \leq i \leq n-1.$$

If $v$ is a vertex of $G$, let $L_v(G)$ be the principal sub-matrix of $L(G)$ formed by deleting the row and column corresponding to vertex $v$. Let $B_n$ be the matrix of order $n$ obtained from $L(P_{n+1})$ by deleting the row and column corresponding to one of end vertices of $P_{n+1}$, and $H_n$ be the matrix of order $n$ obtained from $L(P_{n+2})$ by deleting the rows and columns corresponding to two end vertices of $P_{n+2}$.

**Lemma 2.4** [12]. *Set $\Phi(P_0) = 0$, $\Phi(B_0) = 1$, $\Phi(H_0) = 1$. Then we have*

*(1)* $\Phi(P_{n+1}) = (x-2)\Phi(P_n) - \Phi(P_{n-1})$, $(n \geq 1)$;
*(2)* $x\Phi(B_n) = \Phi(P_{n+1}) + \Phi(P_n)$;
*(3)* $\Phi(P_n) = x\Phi(H_{n-1})$, $(n \geq 1)$;
*(4)* $\Phi(C_n) = \frac{1}{x}\Phi(P_{n+1}) - \frac{1}{x}\Phi(P_{n-1}) + 2(-1)^{n+1}$, $(n \geq 3)$;
*(5)* $\Phi(C_{n+1,n}) = (x-1)\Phi(C_n) - \Phi(P_n)$;
*(6)* $\Phi(P_m)\Phi(P_n) - \Phi(P_{m-1})\Phi(P_{n+1}) = \Phi(P_{m-1})\Phi(P_{n-1}) - \Phi(P_{m-2})\Phi(P_n)$, $(m \geq 2, n \geq 1)$.

**Lemma 2.5** [11]. *Let $G = G_1u : vG_2$ be the graph obtained by joining the vertex $u$ of the graph $G_1$ to the vertex $v$ of the graph $G_2$ with an edge. Then*

$$\Phi(G) = \Phi(G_1)\Phi(G_2) - \Phi(G_1)\Phi(L_v(G_2)) - \Phi(G_2)\Phi(L_u(G_1)).$$

Let $\tau(B_l)$ be the least eigenvalue of $B_l$.

**Lemma 2.6.** *If $k > l \geq 1$, then $\tau(B_l) > \tau(B_k)$. Furthermore, $\tau(B_k) = \alpha(P_{2k+1})$.*

**Proof.** From Lemma 2.1, we have $\tau(B_k) = \alpha(P_{2k+1})$. Thus, we have for $k > l \geq 1$,

$$\tau(B_k) = \alpha(P_{2k+1}) = 4\sin^2\frac{\pi}{4k+2} < 4\sin^2\frac{\pi}{4l+2} = \alpha(P_{2l+1}) = \tau(B_l). \qquad \square$$

The next result gives a relation between the algebraic connectivity and some principal submatrices of $L(G)$ which is attributed to Bapat and Pati [2].

**Lemma 2.7.** *Let $G$ be a connected graph. Let $W$ be a set of vertices of $G$ such that $G - W$ is disconnected. Let $G_1$, $G_2$ be two components of $G - W$ and let $L_1$ and $L_2$ be the principal submatrices of $L(G)$ corresponding to $G_1$ and $G_2$, respectively. Suppose that $\tau(L_1) \leq \tau(L_2)$. Then either $\tau(L_2) > \alpha(G)$ or $\tau(L_1) = \tau(L_2) = \alpha(G)$.*

**Lemma 2.8** [13]. *Let $v_{k+1}v_kv_{k-1}\cdots v_2v_1$ be a path of a graph $G$ such that $d(v_1) = 1$, $d(v_2) = \cdots = d(v_k) = 2$, $d(v_{k+1}) \geq 1$. Suppose that $\lambda \neq 0$ is a Laplacian eigenvalue of $G$ and $X$ is an eigenvector corresponding to $\lambda$. Then we have*

$$X(v_i) = X(v_1)\prod_{j=1}^{i-1}(\lambda_j(B_{i-1}) - \lambda), \quad i = 2, \cdots, k, k+1,$$

*where $\lambda_j(B_{i-1})$ is the $j$-th largest eigenvalue of $B_{i-1}$, $(1 \leq j \leq i-1)$.*

**Lemma 2.9** [12]. *Let $G$ be a connected graph on $n$ vertices with girth $g \geq 3$. Then $\alpha(G) \geq \alpha(C_{n,g})$, where $C_{n,g}$ is the lollipop graph shown in Fig. 1, and the equality holds if and only if $G \cong C_{n,g}$.*

Fallat et al. [5] proved that

**Lemma 2.10** [5]. *For a fixed $n$, suppose that $n \geq \frac{3g-1}{2}$ and $g \geq 4$. Then $\alpha(C_{n,g}) > \alpha(C_{n,g-1})$.*

We generalize the above result as follows.

**Theorem 2.11.** *Suppose that $g \geq 4$. Then $\alpha(C_{n,g}) > \alpha(C_{n,g-1})$.*

**Proof.** From Lemma 2.10, in what follows, we may suppose that $n < \frac{3g-1}{2}$. Consider the Laplacian characteristic polynomials $\Phi(C_{n,g})$ and $\Phi(C_{n,g-1})$ of $C_{n,g}$ and $C_{n,g-1}$, respectively. From Lemma 2.5, we have for $n \geq g+1$,

$$\Phi(C_{n,g}) = \Phi(C_g)\Phi(P_{n-g}) - \Phi(H_{g-1})\Phi(P_{n-g}) - \Phi(C_g)\Phi(B_{n-g-1})$$

and

$$\Phi(C_{n,g-1}) = \Phi(C_{g,g-1})\Phi(P_{n-g}) - \Phi(L_v(C_{g,g-1}))\Phi(P_{n-g}) - \Phi(C_{g,g-1})\Phi(B_{n-g-1}),$$

where $v$ is the pendant vertex of $C_{g,g-1}$.

Thus we have

$$\begin{aligned} &\Phi(C_{n,g}) - \Phi(C_{n,g-1}) \\ &= \Phi(P_{n-g})[\Phi(C_g) - \Phi(H_{g-1}) - \Phi(C_{g,g-1}) + \Phi(L_v(C_{g,g-1}))] \\ &\quad + \Phi(B_{n-g-1})[\Phi(C_{g,g-1}) - \Phi(C_g)]. \end{aligned} \tag{2.1}$$

By a suitable labeling to $C_{g,g-1}$, we have $L_v(C_{g,g-1}) = L(C_{g-1}) + Z_{g-1}$, where $Z_{g-1}$ is the matrix of order $g-1$ with 1 in the position $(g-1, g-1)$ and all the other entries are zero. Then $\Phi(L_v(C_{g,g-1})) = \Phi(C_{g-1}) - \Phi(H_{g-2})$.

By (1), (3)–(5) of Lemma 2.4 and the above equation, we have

$$\begin{aligned} &\Phi(C_g) - \Phi(H_{g-1}) - \Phi(C_{g,g-1}) + \Phi(L_v(C_{g,g-1})) \\ &= \left\{\frac{1}{x}\Phi(P_{g+1}) - \frac{1}{x}\Phi(P_{g-1}) + 2(-1)^{g+1}\right\} \\ &\quad - \frac{1}{x}\Phi(P_g) - \{(x-1)\Phi(C_{g-1}) - \Phi(P_{g-1})\} + \left\{\Phi(C_{g-1}) - \frac{1}{x}\Phi(P_{g-1})\right\} \\ &= \frac{1}{x}\Phi(P_{g+1}) - \frac{2}{x}\Phi(P_{g-1}) + 2(-1)^{g+1} \\ &\quad - \frac{1}{x}\Phi(P_g) + \Phi(P_{g-1}) - (x-2)\left[\frac{1}{x}\Phi(P_g) - \frac{1}{x}\Phi(P_{g-2}) + 2(-1)^g\right] \\ &= \frac{1}{x}[\Phi(P_{g+1}) - (x-2)\Phi(P_g)] - \frac{1}{x}[\Phi(P_g) - (x-2)\Phi(P_{g-1})] \\ &\quad + \frac{(x-2)}{x}\Phi(P_{g-2}) + 2(x-1)(-1)^{g+1} \\ &= -\frac{1}{x}\Phi(P_{g-1}) + \frac{1}{x}\Phi(P_{g-2}) + \frac{x-2}{x}\Phi(P_{g-2}) + 2(x-1)(-1)^{g+1} \\ &= \frac{1}{x}[-\Phi(P_{g-1}) + x\Phi(P_{g-2}) - \Phi(P_{g-2})] + 2(x-1)(-1)^{g+1} \\ &= \frac{1}{x}[\Phi(P_{g-2}) + \Phi(P_{g-3})] + 2(x-1)(-1)^{g+1}. \end{aligned} \tag{2.2}$$

From (1), (4) and (5) of Lemma 2.4, we have

$$\begin{aligned} &\Phi(C_{g,g-1}) - \Phi(C_g) \\ &= (x-1)\Phi(C_{g-1}) - \Phi(P_{g-1}) - \Phi(C_g) \\ &= (x-1)\left[\frac{1}{x}\Phi(P_g) - \frac{1}{x}\Phi(P_{g-2}) + 2(-1)^g\right] - \Phi(P_{g-1}) \\ &\quad - \left\{\frac{1}{x}\Phi(P_{g+1}) - \frac{1}{x}\Phi(P_{g-1}) + 2(-1)^{g+1}\right\} \\ &= \frac{1}{x}[(x-1)\Phi(P_g) - \Phi(P_{g+1})] - \frac{x-1}{x}[\Phi(P_{g-1}) + \Phi(P_{g-2})] + 2x(-1)^g \\ &= \frac{1}{x}[\Phi(P_g) + \Phi(P_{g-1})] - \frac{x-1}{x}[\Phi(P_{g-1}) + \Phi(P_{g-2})] + 2x(-1)^g \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{x}[\Phi(P_g) - (x-2)\Phi(P_{g-1})] - \frac{x-1}{x}\Phi(P_{g-2}) + 2x(-1)^g \\
&= -\frac{1}{x}\Phi(P_{g-2}) - \frac{x-1}{x}\Phi(P_{g-2}) + 2x(-1)^g \\
&= -\Phi(P_{g-2}) - 2x(-1)^{g+1}. 
\end{aligned} \tag{2.3}
$$

By (2) of Lemma 2.4, and substituting Eqs. (2.2) and (2.3) into (2.1), we have

$$
\begin{aligned}
&\Phi(C_{n,g}) - \Phi(C_{n,g-1}) \\
&= \Phi(P_{n-g})\left[\frac{1}{x}\Phi(P_{g-2}) + \frac{1}{x}\Phi(P_{g-3}) + 2(x-1)(-1)^{g+1}\right] \\
&\quad + \left[\frac{1}{x}\Phi(P_{n-g}) + \frac{1}{x}\Phi(P_{n-g-1})\right][-\Phi(P_{g-2}) - 2x(-1)^{g+1}] \\
&= \frac{1}{x}[\Phi(P_{n-g})\Phi(P_{g-3}) - \Phi(P_{n-g-1})\Phi(P_{g-2})] \\
&\quad + 2(-1)^{g+1}[(x-2)\Phi(P_{n-g}) - \Phi(P_{n-g-1})].
\end{aligned}
$$

Since $n < \frac{3g-1}{2}$, $n-g-1 \leq 2(n-g-1) < g-3$. Applying (6) of Lemma 2.4 repeatedly and (1) of Lemma 2.4 to the above equation, we have

$$
\begin{aligned}
&\Phi(C_{n,g}) - \Phi(C_{n,g-1}) \\
&= \Phi(P_{2g-n-2}) + 2(-1)^{g+1}\Phi(P_{n-g+1}), \ (n-g-1 \leq g-3).
\end{aligned} \tag{2.4}
$$

Let $\alpha = \alpha(C_{n,g-1})$. Then Lemma 2.1 implies that $\alpha \leq \lambda_{n-3}(C_{g-1} \bigcup P_{n-g+1}) \leq \alpha(P_{n-g+1})$. Thus

$$
(-1)^{n-g+1}\Phi(P_{n-g+1};\alpha) = \prod_{i=1}^{n-g+1}(\lambda_i(P_{n-g+1}) - \alpha) \leqslant 0. \tag{2.5}
$$

Now, we consider the following two cases:

**Case 1:** Suppose that $\alpha(C_{n,g-1}) \geq \tau(B_{n-g+1})$.

Let $w$ be the vertex of $C_{n,g-1}$ with degree 3. Then $\tau(L_w(C_{n,g-1})) = \min\{\tau(B_{n-g+1}), \tau(H_{g-2})\}$. From Lemmas 2.2, 2.7 and (3) of Lemma 2.4, we have

$$
\alpha = \alpha(C_{n,g-1}) \leq \tau(H_{g-2}) = \alpha(P_{g-1}).
$$

Thus, we have $\tau(B_{n-g+1}) \leq \alpha(P_{g-1})$.

Since $n-g+1 > 0$, $2g-n-2 < g-1$. So we have $\alpha = \alpha(C_{n,g-1}) \leq \alpha(P_{g-1}) < \alpha(P_{2g-n-2})$. Thus

$$
(-1)^{2g-n-2}\Phi(P_{2g-n-2};\alpha) = \prod_{i=1}^{2g-n-2}(\lambda_i(P_{2g-n-2}) - \alpha) < 0. \tag{2.6}
$$

Since $\alpha$ is an eigenvalue of $C_{n,g-1}$, $\Phi(C_{n,g-1};\alpha) = 0$. Thus from Eqs. (2.4)–(2.6), we have

$$
\begin{aligned}
&(\lambda_1(C_{n,g}) - \alpha)(\lambda_2(C_{n,g}) - \alpha)\cdots(\lambda_{n-2}(C_{n,g}) - \alpha)(\lambda_{n-1}(C_{n,g}) - \alpha)\alpha \\
&= (-1)^{n-1}\Phi(C_{n,g};\alpha) \\
&= (-1)^{n-1}[\Phi(C_{n,g};\alpha) - \Phi(C_{n,g-1};\alpha)] \\
&= (-1)^{n-1}[\Phi(P_{2g-n-2};\alpha) + 2(-1)^{g+1}\Phi(P_{n-g+1};\alpha)] \\
&= -(-1)^{2g-n-2}\Phi(P_{2g-n-2};\alpha) - 2(-1)^{n-g+1}\Phi(P_{n-g+1};\alpha) > 0.
\end{aligned} \tag{2.7}
$$

From Lemma 2.3, we have $\alpha = \alpha(C_{n,g-1}) \leq \lambda_{n-2}(C_{n,g})$. Thus from Eq. (2.7), we have

$$
\alpha(C_{n,g}) = \lambda_{n-1}(C_{n,g}) > \alpha = \alpha(C_{n,g-1}).
$$

**Case 2:** Suppose that $\alpha(C_{n,g-1}) < \tau(B_{n-g+1})$.

Note that, by Lemma 2.6 we have $\tau(B_{n-g+1}) \leq \tau(B_1) = \alpha(P_3) = 1$. So we only need to consider $\alpha(C_{n,g}) < 1$. From Lemma 2.6, we immediately obtain that $\alpha(C_{n,g-1}) < \tau(B_{n-g})$. We refer to the labeling of $C_{n,g}$ described in Fig. 1. For convenience, we use $X(j)$ to instead of $X(v_j)$.

Let $X$ be a Fiedler vector of $C_{n,g}$. Without loss of generality, we may assume that $X(1) \geq 0$. Then from Lemma 2.8, we have

$$X(j) \geq 0 \text{ for } j = 1, 2, \ldots, n-g+1.$$

Since $\sum_{j=1}^{n} X(j) = 0$ and $X \neq 0$, there exists the least integer $i$ with $n-g+2 \leq i \leq n$ such that $X(i) < 0$.

From $(D(C_{n,g}) - A(C_{n,g}))X = \alpha(C_{n,g})X$, we have

$$(2 - \alpha(C_{n,g}))X(i) = X(i-1) + X(i+1).$$

Since we are assuming $\alpha(C_{n,g}) < 1$, we have $X(i+1) < X(i) < 0$.

(a) Suppose that $X(i+2) \geq 0$. Let

$$C^1_{n,g-1} = C_{n,g} - v_{i+1}v_{i+2} + v_i v_{i+2}.$$

Then we have

$$\begin{aligned} &\alpha(C_{n,g}) - X^T L(C^1_{n,g-1})X = X^T L(C_{n,g})X - X^T L(C^1_{n,g-1})X \\ &= (X(i+1) - X(i+2))^2 - (X(i) - X(i+2))^2 > 0. \end{aligned} \tag{2.8}$$

By (1.1) we have $\alpha(C_{n,g}) > \alpha(C^1_{n,g-1})$. So from Lemma 2.9 we have

$$\alpha(C_{n,g}) > \alpha(C^1_{n,g-1}) > \alpha(C_{n,g-1}).$$

(b) Suppose that $X(i+1) < X(i+2) < 0$.

Suppose that $X(i+2) \leq X(i)$. Let

$$C^2_{n,g-1} = C_{n,g} - v_i v_{i+1} + v_i v_{i+2}.$$

Then we have

$$\begin{aligned} &\alpha(C_{n,g}) - X^T L(C^2_{n,g-1})X = X^T L(C_{n,g})X - X^T L(C^2_{n,g-1})X \\ &= (X(i) - X(i+1))^2 - (X(i) - X(i+2))^2 \geq 0. \end{aligned}$$

By Eq. (1.1) we have $\alpha(C_{n,g}) > \alpha(C^2_{n,g-1})$. Thus, Lemma 2.9 implies that

$$\alpha(C_{n,g}) \geq \alpha(C^2_{n,g-1}) > \alpha(C_{n,g-1}).$$

Suppose that $X(i+2) > X(i)$. From Eq. (2.8) we have $\alpha(C_{n,g}) - X^T L(C^1_{n,g-1})X > 0$. Therefore, Lemma 2.9 implies that

$$\alpha(C_{n,g}) \geq \alpha(C^1_{n,g-1}) > \alpha(C_{n,g-1}).$$

(c) Suppose that $X(i+2) < X(i+1) < 0$. Note that $X(n-g+1) \geq 0$. By similar argument as above, we can construct a unicyclic graph $C^3_{n,g-1}$ obtained from $C_{n,g}$ by deleting some edge $v_k v_{k+1}$ (or $v_{k+1}v_{k+2}$). And then adding a new edge $v_k v_{k+2}$, where $i \leq k \leq n-g-1$. We will obtain $\alpha(C_{n,g}) \geq \alpha(C^3_{n,g-1})$. From Lemma 2.9, we have

$$\alpha(C_{n,g}) \geq \alpha(C^3_{n,g-1}) > \alpha(C_{n,g-1}).$$

From the above discussions, the proof is completed. □

From Theorem 2.11 and Lemma 2.9, we have the following:

**Corollary 2.12** [12]. *Let $G$ be a connected graph on $n$ vertices with girth $g \geq 3$. Then $\alpha(G) \geq \alpha(C_{n,3})$, and the equality holds if and only if $G \cong C_{n,3}$.*

## References


[1] R.B. Bapat, S. Pati, Algebraic connectivity and the characteristic set of a graph, Linear and Multilinear Algebra 45 (1998) 247–273.
[2] S. Barik, S. Pati, On algebraic and spectral integral variations of graphs, Linear Algebra Appl. 397 (2005) 209–222.
[3] D.M. Cvetković, M. Doob, H. Sachs, Spectra of Graphs – Theory and Applications, VEB Deutscher Verlag d. Wiss., Berlin, 1979, Acad. Press, New York, 1979.
[4] S. Fallat, S. Kirkland, Extremizing algebraic connectivity subject to graph theoretic constraints, Electron. J. Linear Algebra 3 (1998) 48–74.
[5] S. Fallat, S. Kirkland, S. Pati, Minimizing algebraic connectivity over connected graphs with fixed girth, Discrete Math. 254 (2002) 115–142.
[6] M. Fiedler, Algebraic connectivity of graphs, Czechoslovak Math. J. 23 (1973) 298–305.
[7] M. Fiedler, A property of eigenvectors of nonnegative symmetric matrices and its application to graph theory, Czechoslovak Math. J. 25 (1975) 619–633.
[8] R. Grone, R. Merris, Algebraic connectivity of trees, Czechoslovak Math. J. 37 (1987) 660–670.
[9] R. Grone, R. Merris, Ordering trees by algebraic connectivity, Graphs Combin. 6 (1990) 229–237.
[10] R. Grone, R. Merris, V.S. Sunder, The Laplacian spectrum of a graph, SIAM J. Matrix Anal. Appl. 11 (1990) 218–238.
[11] J.-M. Guo, On the second largest Laplacian eigenvalue of trees, Linear Algebra Appl. 404 (2005) 251–261.
[12] J.-M. Guo, A conjecture on the algebraic connectivity of connected graphs with fixed girth, Discrete Math. 308 (2008) 5702–5711.
[13] J.-M. Guo, The algebraic connectivity of graphs under perturbation, Linear Algebra Appl. 433 (2010) 1148–1153.
[14] W.H. Haemers, X.-G. Liu, Y.-P. Zhang, Spectral characterizations of lollipop graphs, Linear Algebra Appl. 428 (2008) 2415–2423.
[15] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, New York, 1985.
[16] S. Kirkland, M. Neumann, B.L. Shader, Characteristic vertices of weighted trees via Perron values, Linear and Multilinear Algebra 40 (1996) 311–325.
[17] S. Kirkland, M. Neumann, Algebraic connectivity of weighted trees under perturbation, Linear and Multilinear Algebra 42 (1997) 187–203.
[18] J. Li, J.-M. Guo, W.C. Shiu, The smallest values of algebraic connectivity for unicyclic graphs, Discrete Appl. Math. 158 (2010) 1633–1643.
[19] J. Li, J-M. Guo, W.C. Shiu, The orderings of bicyclic graphs and connected graphs by algebraic connectivity, Electron. J. Combin. 17 (2010), #R162.
[20] R. Merris, Characteristic vertices of trees, Linear and Multilinear Algebra 22 (1987) 115–131.
[21] R. Merris, Laplacian graph eigenvectors, Linear Algebra Appl. 278 (1998) 221–236.
[22] Y.-P. Zhang, X.G. Liu, B.Y. Zhang, X.R. Yong, The lollipop graph is determined by its Q-spectrum, Discrete Math. 309 (2009) 3364–3369.